Indian Statistical Institute, Bangalore Centre
B.Math. (I Year) : 2012-2013
Semester II : Mid-Semestral Examination
Probability Theory II

6.3.2013 Time: $2\frac{1}{2}$ hours. Maximum Marks : 80

Note: Notation and terminology are understood to be as used in class. The paper carries 82 marks. Any score above 80 will be taken as 80. State clearly the results you are using in your answers.

1. ($9 + 9 + 9 = 27$ marks) Let (X, Y) be a two dimensional absolutely continuous random variable with probability density function
$$\begin{aligned} f(x,y) &= C(y-x)^2, \text{ if } 0 < x < y < 1, \\ &= 0, \text{ otherwise} \end{aligned}$$
where C is a constant.
(i) Find the value of C.
(ii) Find the marginal probability density functions.
(iii) Find the correlation coefficient $Cov(X, Y)$.

2. (15 marks) Let X and Y be independent random variables having identical probability density functions given by
$$f(u) = \frac{1}{2}e^{-|u|}, \quad u \in \mathbb{R}.$$
Find the probability density function of $\frac{Y}{X}$.

3. ($12 + 8 + 5 = 25$ marks) Let $\lambda > 0$ be a known constant. Let (X_1, X_2, X_3) be a 3-dimensional absolutely continuous random variable with probability density function
$$\begin{aligned} f(x_1,x_2,x_3) &= \lambda^3 \exp(-\lambda x_3), \quad \text{if} \quad 0 < x_1 < x_2 < x_3 < \infty, \\ &= 0, \text{ otherwise.} \end{aligned}$$
Let $Y_1 = X_1, Y_2 = X_2 - X_1, Y_3 = X_3 - X_2$.
(i) Find the probability density function of (Y_1, Y_2, Y_3).
(ii) Find the distributions of the one dimensional random variables $Y_i, i = 1, 2, 3$. Are they independent?
(iii) Using (ii) find the distributions of $X_j, j = 1, 2, 3$.

4. (15 marks) Suppose $f(\cdot,\cdot)$ is a bivariate normal density given by

$$f(x,y) \quad = \quad C\exp\{-\frac{1}{2}Q(x,y)\},\ (x,y)\in\mathbb{R}^2,$$

where C is a constant and

$$Q(x,y) \quad = \quad \frac{1}{3}[6(x+1)^2-2(x+1)(y-2)+(y-2)^2],\ (x,y)\in\mathbb{R}^2.$$

Find the constant C, the mean vector and the covariance matrix. Recall that bivariate normal density may be written

$$f(x,y)=$$
$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\{-\frac{1}{2}\frac{\sigma_2^2(x-\mu_1)^2-2\rho\sigma_1\sigma_2(x-\mu_1)(y-\mu_2)+\sigma_1^2(y-\mu_2)^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}\}.$$